Quiz 21 : Continuous Probability Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam.

1 Lightbulbs

Consider n lightbulbs, where the lifetime of each is exponentially distributed with parameter λ.

Hint: Let X_i denote the lifetime of the ith lightbulb. We know that $X_i \sim \textsc{Expo}(\lambda)$.

1. If we use one lightbulb at a time, how long will it take to use all light bulbs?

   Solution: We take the total lifetime X to be $\sum_{i=1}^{n}$. Apply linearity of expectation.

   $$E[X] = E[\sum_{i=1}^{n} X_i] = nE[X_i] = \frac{n}{\lambda}$$

2. How much time can we expect until the first lightbulb burns out?

   Solution: We're computing the minimum of all X_i. We know that the minimum of exponentials, where each is distributed with λ_i is simply another exponential, distributed with the sum of all parameters, $\sum_i \lambda_i$. Thus, we will define a new random variable Z to be the minimum.

   $$Z = \textsc{Min}\{X_1, X_2 \dots X_n\} \sim \textsc{Expo}(n\lambda)$$

   Since Z is exponential, we know its expectation.

   $$E[Z] = \frac{1}{\lambda n}$$

   For the curious, here is a proof for why the minimum of exponentials distributed with λ_i is another exponential distributed with $\sum_i \lambda_i$.

First, we realize that the CDF of an exponential random variable $X \sim \textsc{Expo}(\lambda)$ is $1 - e^{-\lambda t}$. Using this, we know

$$\begin{aligned} P(X \geq x) &= 1 - P(X \leq x) \\ &= e^{-\lambda t} \end{aligned}$$

We call this the `survival function`, as it tells us the probability that we survive until time x. We now re-apply this to Z. Without loss of generality, let us consider just consider two exponentials. We can see that inductively, this generalizes to the minimum of n exponentials.

$$\begin{aligned} P(Z \geq z) &= 1 - P(Z \leq z) \\ &= 1 - P(\textsc{Min}\{X_1, X_2\} \leq z) \\ &= 1 - P(X_1 \leq z, X_2 \leq z) \\ &= 1 - P(X_1 \leq z)P(X_2 \leq z) \\ &= 1 - e^{-\lambda_1 t}e^{-\lambda_2 t} \\ &= 1 - e^{-t(\lambda_1 + \lambda_2)} \end{aligned}$$

We note this is simply the CDF of another exponential distribution with parameter $\lambda_1 + \lambda_2$.

3. How much time can we expect until the ith lightbulb burns out?

Solution: Let us consider the random variable Z_i, which denotes the minimum of the first i lightbulbs to die. We note that the expected lifetime of all i lightbulbs is:

- the amount of time it takes the first lightbulb to die
- plus the time it takes the second lightbulb to die *starting from* the death of the first lightbulb
- plus the time is takes the third lightbulb to die *starting from* the death of the second lightbulb etc.

In other words, we reason about the time until the next lightbulb goes out, *starting from* the time of the last lightbulb to die. By the memoryless property, we note that the past for any given lightbulb does not affect the future. In other words, we simply need to sum the minimum of all n lightbulbs, the minimum of the next $n-1$ lightbulbs, the minimum of the next $n-2$ lightbulbs etc. Given i lightbulbs, we have that

$$
\begin{aligned}
E[Z] &= \frac{1}{n\lambda} + \frac{1}{(n-1)\lambda} + \cdots + \frac{1}{\lambda} \\
&= \sum_{i=0}^{n} \frac{1}{(n-i)\lambda}
\end{aligned}
$$

We must take care to note that $i \leq n$.