EE16B Controls

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Chapter 1

State-Space Model

What do we want? At a high-level, we would like to stabilize some physical system. First, we find the physical model, and somehow mold it to fit the state-space model. Second, place eigenvalues. The following chapters will walk through just how to do this and why it works.

1.1 Linear Model

We see several common features of the state-space model.

- **state**: minimal features needed to describe some snapshot ($\vec{x}$)
- **output**: what you can observe about the system ($\vec{y}$)
- **input**: what we can put into the system (u)

Feedback is modeled as some disturbance in the model. We need bring our system to one of two forms.

1.1.1 Form 1: Discrete

The following models a discrete-time system.

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

1.1.2 Form 2: Continuous

The following models a continuous-time system.

$$\dot{x} = Ax + Bu$$
$$\vec{y} = C\vec{x}$$

We begin by examining *how* we bring systems to this form. Continuous makes most sense for real-world system. Consider this: circuits are also modeled by form 1! If we have a second derivative, we need to introduce a new variable to model it as a first derivative.

Given the system,

$$T\ddot{x} + \dot{x} = k\delta$$

We create a new variable $r = \dot{x}$.

$$T\dot{r} + r = k\delta$$

Let us now place this in matrix form.

$$\begin{bmatrix} \dot{x} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/T \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ k/T \end{bmatrix} \delta$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}$$

1.2 Non-Linear Differential Equation

Let us consider the following system.

$$\dot{\vec{x}} = f(\vec{x}, \vec{y})$$

Now, we need to *linearize* first. In other words, we desire the following form.

$$f(x,u) = f(x_0,u_0) + \frac{\partial f}{\partial x}|_{x_0,u_0}(x - x_0) + \frac{\partial f}{\partial y}|_{x_0,u_0}(u - u_0)$$

There are two steps:

1.2.1 Linearize

We need a point to linearize around, such that x_0, u_0 such that $f(x_0, u_0) = 0$. To simplify calculations, we will linearize around a point that causes $f(x, u) = 0$. Specifically, we will use the operating point; the **operating point** (for the purposes of this class), is defined to be any x_0 such that

$$f(x_0, 0) = 0$$

For the spaceship, note that this only occurs when the spaceship is already horizontal.

1.2.2 Jacobian

The Jacobian is defined to be the following.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_0}{\partial x_{m-1}} \\ \vdots & \ddots & & \vdots \\ \frac{\partial f_{n-1}}{\partial x_0} & \cdots & & \frac{\partial f_{n-1}}{\partial x_{m-1}} \end{bmatrix}$$

1.2.3 Final Form

Once we compute the Jacobian and plug it back into the original form, the Jacobian becomes our A and B. Specifically,

$$A = \frac{\partial f}{\partial x}|_{x_0,u_0}$$

$$B = \frac{\partial f}{\partial y}|_{x_0,u_0}$$

We will then have the final form:

$$\dot{x} = A(x - x_0) + B(u - u_0)$$

$$\boxed{\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}}$$

We now have a form that we easily deal with.

Chapter 2

Characterizing Systems

Let us consider all the characteristics of a system:

- **Stable**: If we leave the system alone, it will reach equilibrium. Consider this to be "staying at the same point," and we can generalize that to be trajectory, velocity etc.
- **Controllable**: Can we bring a system to *any* point that we want? If it is, we can place this system in the "right" place.
- **Observable**: When building a system, examine the current state of the system. State is not an output, however, so we build an observer. The observer is possible if the system is observable.

The ideal system is both controllable and observable. This way, we can build both a controller and observer, so that we can provide feedback that stabilizes the system.

2.1 Stability

2.1.1 Discrete-Time System

How do we know if a system is controllable? Consider the discrete-time system without input.

$$x(t+1) = Ax(t)$$

We additionally know the following, if we recursively plug in $x(t)$.

$$\begin{aligned} x(t) &= Ax(t-1) \\ &= A^2 x(t-2) \\ &= A^3 x(t-3) \\ &\vdots \\ &= A^t x(0) \end{aligned}$$

Let us examine all the eigenvalues of A, where $\vec{v}$ is some eigenvector.

$$x(t) = \lambda^t \vec{v}$$

We want $|\lambda| < 1$. This is because then A then decays. On a side note, what happens if λ is imaginary?

$$\lambda = re^{i\theta}$$

We note that

$$\lambda^t = r^t e^{iwt}$$

This means we're moving around the unit circle. Specifically, r is the rate of decay and w gives the frequency of oscillation.

2.1.2 Continuous-Time System

Again, let us consider a system without input.

$$\dot{x} = Ax$$

The solution has the following general form, where $\vec{v_i}$ is the ith eigenvector, c_i is the ith constant, and λ_i is the ith eigenvalue.

$$x(t) = \sum c_i \vec{v_i} e^{\lambda_i t}$$

What again is the criterion for stability? We know that the eigenvalue must lie in the left half the plane, or in other words, it must be negative.

$$Re[\lambda] < 0$$

2.2 Characterizations

Looking at the original system, let us define:

- $\dot{x}$ is $n \times 1$.
- A is $n \times m$.
- B is $n \times n_i$, where n_i is the number of inputs.
- C is $n_o \times n$, where n_o is the number of outputs.

2.2.1 Controllability

How do we determine controllability? We examine the following matrix.

$$C = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}$$

The system is then controllable if C has rank n.

- If we have a matrix C of rank n, then the matrix is invertible.

- If $n_i = 1$, C is $n \times n$ and invertible. This means the system is automatically controllable.

If we a row column in C that is a linear combination of the previous one, we can stop. In other words, if $A^j B = 0$ or $A^j B = \sum_i a_i A^i B$, we know that we don't need to continue computing $A^i B$. This is important because it will be used in the controllable-canonical form argument.

2.2.2 Observability

We consider the following matrix.

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

All that we said for controllability also holds.

- If O has rank n, then the matrix is invertible.
- If $n_o = 1$, O is $n \times n$ and invertible.

This means the system is automatically observable.

We need a matrix O of rank n. We can also end computation if we get a 0 or linear combination of the previous row vectors.

Note the difference between having rank n and being invertible.

2.3 Controlling the System

How can we add inputs, such that feedback stabilizes the system?

We add to our definition of the system.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u(t) = -Fx$$

The third item $-Fx$ is the control law. Now, we plug in.

$$\begin{aligned} \dot{x} &= Ax + B(-Fx) \\ &= Ax - BFx \\ &= (A - BF)x \end{aligned}$$

So, we pick an F such that $\dot{x} = (A - BF)x$ is stable. We want to choose F such that $A - BF$ is stable. In other words, we would like to *place eigenvalues* in $A - BF$ via F. Take F to be the following.

$$F = \begin{bmatrix} f_0 & f_1 & \dots & f_{n-1} \end{bmatrix}$$

2.3.1 The Algorithm

Recall controllable canonical form.

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & & \alpha_{n-1} \end{bmatrix}$$

We can now reconsider the original system, and write it out explicitly. We previously derived the final form $\dot{x} = (A - BF)x$, so we are only plugging matrices here.

$$\dot{x} = Ax + Bu$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & & \alpha_{n-1} \end{bmatrix} x - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} f_0 & f_1 & \dots & f_{n-1} \end{bmatrix}$$

$$= (\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & & \alpha_{n-1} \end{bmatrix} + \begin{bmatrix} 0 & \dots & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & \\ f_0 & f_1 & \dots & f_{n-1} \end{bmatrix})x$$

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ (\alpha_0 + f_0) & (\alpha_1 + f_1) & \dots & & (\alpha_{n-1} - f_{n-1}) \end{bmatrix}$$

We take the characteristic polynomial using $det(\lambda I - A)$. So, depending on the parity of the matrix dimension, $det(A - \lambda I)$ may either give us positive or negative eigenvalues. So, we take the former instead: $det(\lambda I - A)$, which has the following general form.

$$\alpha_0 + \alpha_1 \lambda + \dots + \alpha_{n-1} \lambda^{n-1} + \lambda^n$$

The roots of this characteristic polynomial become the eigenvalues. So, we can place eigenvalues using this characteristic polynomial. Specifically, consider the set of eigenvalues we desire:

$$(\lambda - \lambda_0)(\lambda - \lambda_1) \dots (\lambda - \lambda_{n-1})$$
$$\beta_0 + \beta_1 \lambda + \dots + \beta_{n-1} \lambda^{n-1} + \lambda^n$$

Thus, we can set all β_i to α_i, giving us a system of linear equation.

$$\alpha_0 + f_0 = \beta_0$$
$$\alpha_1 + f_1 = \beta_1$$
$$\vdots$$

$$\alpha_{n-1} + f_{n-1} = \beta_1$$

We have a linear solvable linear system, with a unique solution. Thus, this gives us solutions for the f_i. Then, we plug back into F, and we have satisfied the control law that will stabilize our system.

2.3.2 Characteristic Polynomial

As an aside, how did we get the characteristic polynomial? Take some eigenvector $\vec{v}$ and its corresponding eigenvalue λ.

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & & \alpha_{n-1} \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} \lambda v_0 \\ \lambda v_1 \\ \vdots \\ \lambda v_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ \sum_i \alpha_i v_i \end{bmatrix}$$

Using this system, we get

$$v_1 = \lambda v_0$$
$$v_2 = \lambda v_1 = \lambda^2 v_0$$
$$\vdots$$
$$v_{n-1} = \lambda^{n-1} v_0$$
$$\lambda v_{n-1} = -\sum \alpha_i v_i$$

Now, plug in v_{n-1} from the second-to-last equation into the last equation.

$$\lambda^n v_0 = -\sum_i \alpha_i \lambda^{i-1} v_0$$

Now, we move the summation to the left and divide by v_0.

$$\lambda^n + \sum_i \alpha_i \lambda^i = 0$$

$$\alpha_0 + \alpha_1\lambda + \ldots + \alpha_{n-1}\lambda^{n-1} + \lambda^n = 0$$

Chapter 3

Proof of Controllable Canonical Form

Consider the system.

$$\dot{x} = Ax + Bu$$

We have some desired system

$$\dot{z} = \tilde{A}z + \tilde{B}u$$

We will define $z = T^{-1}x$, meaning $x = Tz$.

$$\begin{aligned} \dot{Tz} &= ATz + Bu \\ T\dot{z} &= ATz + Bu \\ \dot{z} &= T^{-1}ATz + T^{-1}Bu \end{aligned}$$

This means that if $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$. So, we have the system.

$$\dot{z} = \tilde{A}z + \tilde{B}u$$

Our controllability matrix becomes the following.

$$\begin{aligned} \tilde{C} &= \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots \tilde{A}^{n-1}\tilde{B} \end{bmatrix} \\ &= \begin{bmatrix} T^{-1}B & T^{-1}ATT^{-1}B & \dots & T^{-1}A^{n-1}B \end{bmatrix} \\ &= T^{-1} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \\ &= T^{-1}C \end{aligned}$$

Finally, we get that $\tilde{C} = T^{-1}C$, so

$$T = \tilde{C}^{-1}C$$

We now want to verify that T takes us to controllable-canonical form. Let us discard all other assumptions; we assume that $T = \tilde{C}^{-1}C$, and check that T achieves what we assume it to do.

We know we have two systems, with controllability matrices C and $\tilde{C}$, where $T = \tilde{C}^{-1}C$.

$$\tilde{A} = T^{-1}AT$$
$$\tilde{B} = TB$$

We now compute A.

$$\begin{aligned} A &= T\tilde{A}T^{-1} \\ A &= C\tilde{C}^{-1}\tilde{A}\tilde{C}C^{-1} \\ C^{-1}AC &= \tilde{C}^{-1}\tilde{A}\tilde{C} \end{aligned}$$

We now have to show that these two are equivalent. We consider some arbitrarily C' and A', where C' is the controllability matrix for a system defined by A'.

$$\begin{aligned} &C'^{-1}A'C'\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \\ &= C'^{-1}A'A'^{k}B' \\ &= C'^{-1}A'^{k+1}B' \end{aligned}$$

We now wish to reduce the above equation $C^{-1}A^{k+1}B$.

$$\begin{aligned} C'^{-1}A'^{k+1}B' &= x \\ A'^{k+1}B' &= C'x \end{aligned}$$

We need to recover the structure of $A'^{k+1}B'$. Call this matrix M. First, if $k < n-1$, $M = A'^{k+1}B'$ exists in C', since C' is $\begin{bmatrix} B' & A'B' & \dots & A'^{n-1}B' \end{bmatrix}$. This means up to $k < n-1$, M contains the identity matrix, which simply recovers vectors from C'. However, if $k = n-1$, then $A'^{k+1}B' = A'^{n}B'$, which does not exist in C'. Thus, it is some unspecified linear combination of vectors. This thus means the final form of M is the transpose of $\tilde{A}$!

We know that both $C^{-1}AC$ and $\tilde{C}^{-1}\tilde{A}\tilde{C}$ equal to $\tilde{A}^{T}$ and thus share the same characteristic polynomial.

Now, we wish to show $T^{-1}B$, so we know that $\tilde{B}$ moves to the correct form.

$$\begin{aligned} &T^{-1}B \\ &= \tilde{C}C^{-1}B \\ &= \tilde{C}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

$$= \tilde{B}$$

We've now shown that T takes us to controllable-canonical form. In the new system, we can similarly derive a $\tilde{F}$ that we can control.

$$\begin{aligned} u &= -\tilde{F}z \\ &= -\tilde{F}T^{-1}z \end{aligned}$$

Thus, $F = \tilde{F}T^{-1}$, and we have successfully stabilized our system.

Chapter 4

Observer

Insert diagram

We can't observe the state however. So, we construct an observer, which gives us an $\hat{x}$, an estimate for the state x. The feedback is then constructed using this $\hat{x}$ and fed back into the original system as input.

Insert more complicated digram

Let us construct a simple observer. What is wrong with it?

$$\dot{\hat{x}} = A\hat{x} + Bu$$
$$\hat{y} = C\hat{x}$$

Quite simply, it doesn't account for y. Let us consider the difference with the original system.

$$\dot{\hat{x}} - \dot{x} = A(\hat{x} - x)$$
$$\dot{e} = Ae$$

The issue is that $\dot{e}$ may explode. Thus, we should apply a feedback so that it brings $\dot{e}$ to 0. Specifically, we'll take the error of our system and account for that difference in our system. We now modify our original model.

$$\dot{\hat{x}} = A\hat{x} + Bu - L(\hat{y} - y)$$
$$\hat{y} = C\hat{x}$$

We continue to expand the first equation.

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - Cx)$$
$$\dot{\hat{x}} = A\hat{x} + Bu - LC(\hat{x} - x)$$

Subtract $\dot{x}$ from $\dot{\hat{x}}$.

$$\dot{\hat{x}} - \dot{x} = A(\hat{x} - x) - LC(\hat{x} - x)$$
$$\dot{e} = (A - LC)e$$

Again, we have a matrix $A - LC$, so we can place eigenvalues to bring the error to 0, in theory. However, it is not in the correct form. We know how to place eigenvalues for matrices of the form $A - BF$.

Consider a change of basis, into the **dual system**, defined by

$$\dot{z} = A^T z + C^T v$$

Find L^T, such that

$$\dot{z} = (A^T - C^T L^T)z$$

Since the transpose of a matrix has the same eigenvalues as itself, we know $A^T - C^T L^T$ has the same eigenvalues as $A - LC$. Since L^T is in the correct position, we can place eigenvalues. Once we do that, we have a stable system where $A - LC$ asymptotically approaches 0. However, if we set all eigenvalues to 0, then we know that $A - LC$ is nilpotent, guaranteeing that the $\dot{e}$ is 0 in finite time. Using the dual system, we will now rewrite our model.

$$\dot{x} = Ax - BF\hat{x}$$

Since, $\dot{e} = \hat{x} - x$, we know that $\hat{x} = \dot{e} + x$.

$$\begin{aligned} \dot{x} &= Ax - BFx - BFe \\ &= (A - BF)x - BFe \end{aligned}$$

Writing this with our previous equation, we know have the following system.

$$\begin{aligned} \dot{x} &= (A - BF)x - BFe \\ \dot{e} &= (A - LC)e \end{aligned}$$

We finally rewrite this in matrix form.

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BF & -BF \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

This gives us a single model.