Quiz 23 Solutions

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Tuesday, April 19, 2016

This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam. Today, we will walk through conditional expectation using ideas from dilution and mixing in lecture.

In this section, we are dealing with two new, major ideas: (1) treating conditional expectations as "functions" and (2) recursively defined states.

First, note that $E[Y|X]$, means that *given* a particular Y, we have information about X. In other words, $Y = f(X)$! This generalizes to conditioned joints as well. So, $E[Y|X_1, X_2, X_3...]$ gives us $Y = f(X_1, X_2, X_3...)$.

Thus, using conditional expectation, we can solve recursively defined states. Specifically, we can solve systems of the following form.

$$X(t+1) = \alpha X(t)$$

Translated into a conditional expectation, we simply solve $E[X_{n+1}|X_n]$ given some $E[X_1]$. In fact, we will see that we can even solve systems of the form.

$$X(t+1) = \alpha X(t) + \beta$$

1 Dilution

It is now year 3000 where watermelon is a sacred fruit. Everyone receives N watermelons at birth. However, citizens of this future must participate in the watermelon ceremonies, annually. At this ritual, citizens can choose to pick 1 melon at random, to replace with cantaloupes.

1. Given that a citizen has m watermelons at the nth year, what are all the possible number of watermelons that this citizen can have in the $n+1$th year, and what is the probability that each of these situations occur?

Solution: $X_{n+1} = \begin{cases} m & w.p. 1 - \frac{m}{N} \\ m-1 & w.p. \frac{m}{N} \end{cases}$

In the first case, our number of watermelons does not change. This only occurs if our pick is a cantaloupe. Since there are m watermelons, there are $N - m$ cantaloupes. Thus, the probability of picking a single cantaloupe is $\frac{N-m}{N} = 1 - \frac{m}{N}$.

The second case falls out, as there are m watermelons, making $\frac{m}{N}$.

2. Let us suppose (for this part only) that a citizen now picks **two** watermelons at random, at this ritual. Given that a citizen has m watermelons at the nth year, what are all the possible number of watermelons that this citizen can have in the $n+1$th year, and what is the probability that each of these situations occur?

   Solution: $X_{n+1} = \begin{cases} m & w.p. \frac{(N-m)(N-m-1)}{N(N-1)} \\ m-1 & w.p. \frac{2m(N-m)}{N(N-1)} \\ m-2 & w.p. \frac{m(m-1)}{N(N-1)} \end{cases}$

   In the first case, our number of watermelons does not change. This only occurs if both of our picks are cantaloupes. Since there are m watermelons, there are $N - m$ cantaloupes. Thus, the probability of picking a single cantaloupe is $\frac{N-m}{N}$. The probability of picking two cantaloupes is $\frac{(N-m)(N-m-1)}{N(N-1)}$.

   Likewise, if the number of watermelons decreases by 1, we have chosen one watermelon and one cantaloupe. This means we either chose the watermelon second and the cantaloupe first $\frac{N-m}{N}\frac{m}{N-1}$, or we chose the watermelon first and the cantaloupe second $\frac{m}{N}\frac{N-m}{N-1}$. Summed together, we have that the probability of one watermelon and one cantaloupe is $\frac{2m(N-m)}{N(N-1)}$.

   Finally, the probability of picking two watermelons is $\frac{m(m-1)}{N(N-1)}$.

3. Again, let us consider the original scenario, where each citizen picks only 1 melon at random at the ritual. Given that a citizen has m watermelons at the nth year, how many watermelons will a citizen then have in year $n+1$, on average?

   Solution: $X_n(1 - \frac{1}{N})$

   We are effectively computing $E[X_{n+1}|X_n = m]$. We already considered all possible values of X_{n+1} with their respective probabilities. So,

$$E[X_{n+1}|X_n = m] = \sum_x E[X_{n+1} = x|X_n = m]$$
$$= \sum_x xPr[X_{n+1} = x|X_n = m]$$
$$= m(1 - \frac{m}{N}) + (m-1)\frac{m}{N}$$
$$= m - \frac{m^2}{N} + \frac{m^2}{N} - \frac{m}{N}$$
$$= m - \frac{m}{N}$$
$$= m(1 - \frac{1}{N})$$

Since $X_n = m$, we substitute it in.

$$= X_n(1 - \frac{m}{N})$$

4. After n years, compute the average number of watermelons a particular citizen will have left.

   Solution: $(1 - \frac{1}{N})^{n-1}N$

   We are now computing $E[X_n]$. First, we note that the law of total expectation allows us to conclude the following.

$$E[X_{n+1}] = E[E[X_{n+1}|X_n]]$$
$$= E[X_n(1 - \frac{1}{N})]$$
$$= (1 - \frac{1}{N})E[X_n]$$

   We have the following relationship.

$$E[X_n] = (1 - \frac{1}{N})E[X_{n-1}]$$

   Since $E[X_n]$ is recursively defined, we see that the constant in fromt of $E[X_{n-1}]$ will simply be multiplied repeatedly. Thus, we can express this in terms of $E[X_1]$.

$$E[X_n] = (1 - \frac{1}{N})^{n-1}E[X_1]$$

   Finally, we note that we began with N watermelons, so $E[X_1] = N$.

$$E[X_n] = (1 - \frac{1}{N})^{n-1} N$$

5. If all citizens begin with 100 watermelons, after how many years will citizens in this society end up with 99 cantaloupes?

   Solution: Just plug into our expression for $E[X_n]$. 99 cantaloupes means 1 watermelon. Thus, we are solving for $E[X_n] = 1$.

$$E[X_n] = (1 - \frac{1}{100})^{n-1} 100 = 1$$
$$(\frac{99}{100})^{n-1} = \frac{1}{100}$$
$$\log (\frac{99}{100})^{n-1} = \log \frac{1}{100}$$

   Using the log rule, $\log a^n = n \log a$, we get:

$$(n-1) \log \frac{99}{100} = \log \frac{1}{100}$$
$$n - 1 = \frac{\log \frac{1}{100}}{\log \frac{99}{100}}$$
$$n = 1 + \frac{\log \frac{1}{100}}{\log \frac{99}{100}}$$
$$n \sim 460$$

   It will take approximately 100 years. As it turns out, the number of watermelons we start with is approximately the number of years we will last.