Quiz 18 Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam. In this quiz, we will walk through identifying identifying random variables.

Solution: You should recognize that a problem asking for "how many [thing]s satisfy [condition]" is a binomial distribution, which immediately means two things:

1. Our random variable is *most likely* an indicator, with 1 for success on the ith trial.
2. As with all indicator variables, $E[X] = p$, where p is the probability of success on the ith trial.

In case you're confused why $E[X] = p$ for indicator variables, remember the following.

$$E[X] = [X = 1] + E[X = 0]$$
$$= P[X = 1] * 1 + P[X = 0] * 0$$
$$= p * 1 + (1 - p) * 0$$
$$= p$$

As intimidating as this may be, you should find that this instinctual response ("how many" $\rightarrow$ binomial distribution $\rightarrow$ indicators) will reduce the problem to computing the probability of success on the ith trial.

1 Identifying Random Variables

1. Paul and Nathan thoroughly enjoy oranges; however, Nathan doesn't like peeling oranges. Everytime Paul finishes peeling an orange, Nathan will

with probability p successfully steal Paul's peeled orange. If Paul peels n oranges, how many oranges will he eat successfully?

Solution: Let $X = \sum_i X_i$ be the number of oranges Paul eats, where X_i is 1 if Paul eats an orange on the ith trial.

We now that $P[X_i]$ is the probability that Nathan does not steal an orange, making $P[X_i] = 1 - p$

$$\begin{aligned} E[X] &= E[\sum_i X_i] \\ &= \sum_i E[X_i] \\ &= nE[X_i] \\ &= n(1-p) \end{aligned}$$

This is exactly the formula for a binomial distribution's mean.

2. Angie goes to the music building daily to practice singing. Each day, she chooses one of n pieces at random. Given some $m < n$ of the pieces are arias, let us impose an absolute ordering on these arias. If Angie practices for k days, how many times will Angie practice all arias in sequential order, without repeating an aria? (Note that this means Angie will necessarily spend m days, practicing one aria a day, to finish one sequence.)

   Solution: We see "how many". Our instinctual response is "binomial distribution" and "pick an indicator variable". However, we need to be careful.

   We define X to be the total number of sequences and $X = \sum_i X_i$ where X_i is be 1 iff Angie begins a sequence on day i. This means that the last day Angie can begin a sequence is $k - m + 1$. Thus, we actually consider $k - m + 1$ trials.

   Now, we compute the probability that, of m trials, Angie picks exactly the right m arias in sequential order. Thus, the probability for a particular X_i is $\frac{1}{n^m}$.

$$\begin{aligned} E[X] &= E[\sum_i X_i] \\ &= \sum_i E[X_i] \\ &= (k-m+1)E[X_i] \end{aligned}$$

$$= (k - m + 1)\frac{1}{n^m}$$