Quiz 14 Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam. In this quiz, we will walk through stars and bars variations of counting, with problems of increasing difficulty.

1 Stars and Bars (with Ice Cream again)

Assume that sprinkles are indistinguishable from one another.

As warmup, complete question 3 from the previous quiz, for the most basic form of stars and bars.

1. How many ways are there to sprinkle 10 sprinkles on 3 scoops, such that the first scoop gets at least 5 pieces?

   Solution: $\binom{7}{2}$. This reduces to a second stars and bars problem. Simply, we first give the first scoop 5 pieces. Why do this? This means that regardless of however many *additional* pieces we distribute it, the first scoop will have at least 5 pieces.

   We are then left with a sub-stars-and-bars problem, with 10-5=5 sprinkles and 3 scoops. We proceed as usual, noting this is 5 stars and 2 bars.

2. Assume that each scoop can only hold a maximum of 8 pieces. How many ways are there to sprinkle 10 sprinkles on 3 scoops?

   Solution: $\binom{12}{2} - \binom{3}{1}\binom{2}{1} - \binom{3}{1}$

   First, we count all the possible ways to distribute 10 oreo sprinkles among 3 scoops. This is the answer to the last quiz, problem 3: $\binom{12}{2}$

   Then, we count the number of invalid combinations. The only invalid combinations are when a scoop has 9 or more sprinkles. We consider each case:

(a) One scoop has 9 sprinkles. There are $\binom{3}{1}$ ways to pick this one scoop with 9 sprinkles. Then, there are two other scoops to pick from, to give the final sprinkle, making $\binom{2}{1}$ ways to "distribute" the last sprinkle.

(b) One scoop has 10 sprinkles. There are $\binom{3}{1}$ ways to pick this one scoop. There are no more sprinkles for the other scoops.

Thus, we take all combinations and then substract both invalid combinations. We note that the invalid combinations are mutually exclusive.

3. Assume that each scoop can only hold a maximum of 8 pieces and a minimum of 2. How many ways are there to sprinkle 14 sprinkles on our 3 scoops of ice cream?

Solution: $\binom{10}{2} - \binom{3}{1}\binom{2}{1} - \binom{3}{1}$

Given the first problem in the quiz, we know that we can reduce the problem to a sub-stars-and-bars problem. We first distribute 2 sprinkles to each scoop, guaranteeing that each scoop will have at least 2 sprinkles distributed to it.

Then, we count all the possible ways to distribute 8 sprinkles among 3 scoops. This is - by stars and bars - $\binom{10}{2}$.

Then, we count the number of invalid combinations. One set of invalid combinations is when a scoop has 9 or more sprinkles; this was found in the last problem. $\binom{3}{1}\binom{2}{1} + \binom{3}{1}$ Note that we can directly apply the answer from the previous part, only because these values are all less than the number of sprinkles we have left. If these accounted for cases with more sprinkles than are remaining, we would need to apply inclusion-exclusion.