Quiz 6 Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam. "Intuition Practice" might be tricky; watch out for subtleties. "Proofs" will be challenging to start; develop an arsenal of *approaches* to starting a problem.

1 Proofs

1. **Strongly-connected Components** A strongly-connected component of a graph is a set of vertices S where each pair of vertices $(u, v) \in S$ is connected by a path. Prove that a connected graph with directed edges cannot contain SCCs if there are fewer than $|V|$ edges.

   This means the graph G has at most $|V| - 1$ edges, which by definition is a tree. Since a tree can be sorted topologically, we have that no forward edges are accompanied by a back edge. In other words, no child has a directed edge back to a parent. Thus, no SCCs can exist.

2. **Hypertriangles**

   Let us define a "hypertriangle, where the 1-dimensional hypertriangle is the triangle. Hypertriangles are defined recursively, as hypercubes are, except each n-dimensional hypertriangle is defined using 3 (n-1)-dimensional hypertriangles.

   (a) Prove that the n-dimensional hypertriangle consists of 3^n vertices.

   Each dimension multiplies the number of vertices by 3. We see that since the 1-dimensional hypertriangle contains 3 points, each dimension will contain 3^n vertices.

   (b) Prove that the degree of all vertices in the hypertriangle are the same.

   It's easier to prove a stronger statement: the degree of all vertices in the n-dimensional hypertriangle is $2n$. Looking at the ternary representation of the hypertriangle, there are exactly n digits we can change, where each digit has 2 options that differ from the original digit.

   (c) Prove that the parity of each vertex's degree is opposite the parity of the dimension n.

   In part (c), we showed that each vertex's degree is $2n$. This is, by construction, even. We can additionally note that with each increase in dimension, we add two degrees to each vertex. We could formalize the proof inductively.

(d) Find the number of edges in an n-dimensional hypertriangle and justify your answer.

We first define a recursive formula, using a function $E(i)$ which will return the number of edges for the ith dimensional hypertriangle. We note that each hypertriangle has three component hypertriangles.

$$E(n) = 3E(n-1) + ?$$

Since each vertex in the old hypertriangles must connect to a new component, we add the number of vertices.

$$E(n) = 3E(n-1) + 3^n$$

If viewed as a tree, we note that at the ith level, we have 3^i nodes, where the given n' is now actually $n - i$. Knowing this, we have the work done by each node is $3^{n'} = 3^{n-i}$. Additionally, the tree has height n, as the problem size is decreased by 1 each time. To compute $E(n)$, then:

$$\begin{aligned} E(n) &= \sum_{i=0}^{n} 3^i 3^{n-i} \\ &= \sum_{i=0}^{n} 3^n \\ &= n3^n \end{aligned}$$