Quiz 2 Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam. "Intuition Practice" might be tricky; watch out for subtleties. "Proofs" will be challenging to start; develop an arsenal of *approaches* to starting a problem.

1 Proofs

1. **To Induction or Not**

   Prove that $\forall i, a \in \mathbb{N}$ where $x_1 = 1 \wedge x_i = i^2 + i + a - x_{i-1}$, the parity of x_i is opposite the parity of a. *Note: Parity is whether a number is even or odd.*

   How do I start?
   As with all induction-type problems, begin by doing mindless work. Verify that the base case is correct, write the induction hypothesis, and state what you are attempting to prove.

   In this problem, we note a simplifying step however. This step is optional. There are two options:

   (a) We run through induction twice. Once, with a as odd and a second time with a as even.

   (b) We first note that for x_i's parity to be opposite that of a, $\forall i \in \mathbb{N}, i^2+i+x_{i-1}$ must be odd. If this is confusing, see the last paragraph of this proof.

   We will prove the second statement instead, that all $i^2 + i + x_i$ are odd.

   Base Case
   The $i = 1$ case is already given. Thus, for our base case, we will consider $i = 2$. $x_2 = 2^2 + 2 - 1 = 5$, which is odd. Our base case has been proven.

   Inductive Hypothesis
   We now assume that for $i = k$ is satisfied. In other words, we are assuming that $x_k = k^2 + k - x_{k-1}$ is odd.

   Inductive Step
   We will now prove that x_{k+1} is also odd. To begin, let us write x_{k+1}.

$$x_{k+1} = (k+1)^2 + k - x_k$$

There is next to nothing we can do to manipulate this statement, algebraically. We know that x_k is odd, and we have its value as a function of k and x_{k-1}. Plugging it in gives us no information. Have we hit a wall? Not exactly.

We can now consider all possibilities for $k + 1$. Since we are considering parity in this proof, we should consider the cases where (a) $k + 1$ is odd and (b) where $k + 1$ is even:

(a) If $k+1$ is odd, we know that $x_{k+1} = odd^2+odd+x_k = odd+odd+x_k = even + x_k$.

(b) If $k + 1$ is even, we know that $x_{k+1} = even^2 + even + x_k = even + even + x_k = even + x_k$.

Regardless of $k+1$'s parity, $x_{k+1} = even+x_k$. By the inductive hypothesis, we know that x_k is odd, and since an even plus an odd is always odd, x_{k+1} is odd.

By induction, we have proved that $\forall i \in \mathbb{N}, i^2 + i + x_{i-1}$ is odd. Since $even + odd = odd$ and $odd + odd = even$, the parity of a will be opposite that of the parity of all x_i.

2. **Look and Say**

   The look-and-say sequence is a sequence of numbers, where the next number is a translation into English of the previous term read aloud. The sequence begins with 1. For example, "11" would read "two 1s", which gives us the next term: "21". The next number after that is "1211".

   (a) Prove that all numbers in the look-and-say sequence end in 1.

   How do I start?

   As with all induction-type problems, begin by doing mindless work. Verify that the base case is correct, write the induction hypothesis, and state what you are attempting to prove.

   Base Case

   The sequence begins with 1. This is trivially true.

   Inductive Hypothesis

   We assume that the kth term ends with 1.

   Inductive Step

   Now, we need to prove that the $k + 1$st term ends with 1. If the kth term ends with 1, then the two rightmost digits of the $k + 1$st would be of the form [quantity]1. This makes the last digit of the $k + 1$st term 1.

   (b) Prove that no number in the look-and-say sequence will contain a digit greater than 3.

   How do I start?

   There is little we can say about the *lack* of a digit greater than 3. However, the negation allows us to write a mathematical expression. Consequently, we know that we should with the negation of the proposition, either contraposition or contradiction.

   Proof

   Assume for contradiction that there exists a number 4 in term x_n.

This means that the previous term x_{n-1} contains a sequence of 4 consecutive integers of the same value. Let us assume without loss of generality that this value was 1. This implies that the previous term before that x_{n-2} contained "one 1 one 1", or "11". However, we know that the look-and-say sequence would have generated "21". Contradiction.