Quiz 1 Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam. "Intuition Practice" might be tricky; watch out for subtleties. "Proofs" will be challenging to start; develop an arsenal of *approaches* to starting a problem.

1 Intuition Practice

For the following, we assume $\forall x \in \mathbb{Z}, A(x) \wedge B(x) \implies \exists y \in \mathbb{Z}, C(x, y)$. Your options are True or False.

1. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, C(x, y)$

   False. If $A(x) \wedge B(x)$ is not satisfied, we know nothing about $C(x, y)$. It is possible that $P = \forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg C(x, y)$ if $\neg A(x) \wedge B(x)$.

2. $\forall x \in \mathbb{Z}, A(x) \wedge B(x) \iff \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, C(x, y)$

   False. The forward implication is True. Even though there is a second universal quantifier $\forall x \in \mathbb{Z}$, both still specify the same universe, making it equivalent to the P. The reverse implication is false, however. Intuitively, it is the converse of the original proposition. A counterexample is any set of propositions A, B, C that create a vacuous truth in P.

3. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg A(x) \vee \neg B(x) \vee C(x, y)$

   True. This is just the expanded form of an implication $(P \implies Q) \iff (\neg P \vee Q)$

2 Proofs

1. Prove or disprove that $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^3 + 3n^2 + 2n = 3k$.

   We see that if split, $3n^2$ allows us to factor out n from the first and third terms.

   $$n^3 + n^2 + 2n^2 + 2n$$

   Group terms.

$$= n^2 \cdot (n+1) + 2n \cdot (n+1)$$
$$= (n+1) \cdot (n^2 + 2n)$$

Notice that we can factor out an n from the second term.

$$= (n+1) \cdot n \cdot (n+2)$$
$$= n \cdot (n+1) \cdot (n+2)$$

This equation represents a sequence of three consecutive integers multiplied together. Among any three consecutive integers, one must be a multiple of 3. As a consequence, the sequence can naturally be written in the form of $3k$.

2. Consider 2×2 "L"-shaped tiles. Prove that if a board has dimension $w \times h$, such that $w, h \in \mathbb{Z}, (\forall k \in \mathbb{Z}, h \neq 3k) \land (\exists k \in \mathbb{Z}, \forall l \in \mathbb{Z}, w = 2k \neq 3l)$, the board has no perfect tiling. Define a "perfect tiling" to be a configuration of L-shaped tiles, where no block on the board is left uncovered and each slot contains at most one tile.

Preamble

We will use a proof by contradiction. Note that the original statement is $P \implies Q$, where P and Q are the following.

$P =$ board has dimension $w \times h$ s.t. $(\forall k \in \mathbb{Z}, h \neq 3k) \land (\exists k \in \mathbb{Z}, \forall l \in \mathbb{Z}, w = 2k \neq 3l)$

$Q =$ board has no perfect tiling

In a contradiction, we assume a statement P' and show $R \land \neg R$. Note, however, that we do *not* assume $\neg P$, where P is the first clause in our implication. Instead we must negate *all* of $P' \iff (P \implies Q)$.

$$\neg(P \implies Q)$$
$$\neg(\neg P \lor Q)$$
$$P \land \neg Q$$

So, our assumption for the the contradiction must suggest that a board with the aforementioned dimensions can have a perfect tiling.

Proof

Assume for contradiction a perfect tiling exists for boards of dimension $3 \times (3k+1)$ and $2 \times (3k+2)$.

With boards of $2 \times (3k+1)$, the total area is $6k+2$. Seeing as each tile is composed of 3 units, we will always have $(6k+2)\%3 = 2$ uncovered blocks. Contradiction. Adding an additional tile would give us $6k+3$ total units for a $6k+2$ un.^{2} board. By the pigeonhole principle, one slot would contain two tiles. Contradiction.

With boards of $2 \times 3k + 2$, the total area is $6k + 4$. We would have $(6k + 4)\%3 = 1$ uncovered block. Contradiction. By the pigeonhole principle, adding another tile would lead one block to contain two tiles. Contradiction.

Our assumption leads to a contradiction, and thus, the original statement holds.