Quiz 22 : Continuous Probability II Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam.

1 More Lightbulbs

Consider n lightbulbs, where the lifetime of each is exponentially distributed with parameter λ.

Hint: Let X_i denote the lifetime of the ith lightbulb. We know that $X_i \sim \textsc{Expo}(\lambda)$.

1. Compute the PDF of $Z = X_1 + X_2$. We are re-deriving the *Erlang distribution*.

   Solution: We can use a convolution.

$$\begin{aligned} f_Z(z) &= \int f_{X_1}(z-x) f_{X_2}(x) dx \\ &= \int_0^\infty \lambda^2 e^{-\lambda(z-x)} e^{-\lambda x} dx \\ &= \lambda^2 \int_0^z e^{-\lambda z} dx \\ &= \lambda^2 z e^{-\lambda z} \end{aligned}$$

2. Compute the PDF of $Y = X_1 + X_2 + X_3$. Do you notice a pattern?

   Solution: We can take $f_Z(z)$ and convolve it with X_3.

$$
\begin{aligned}
f_Y(y) &= \int f_Z(y-x) f_{X_3}(x) dx \\
&= \int_0^\infty \lambda^3 (y-x) e^{-\lambda(y-x)} e^{-\lambda x} dx \\
&= \lambda^3 e^{-\lambda z} \int_0^y (y-x) dx \\
&= \lambda^3 e^{-\lambda z} (yx - \frac{x^2}{2})_0^y \\
&= \lambda^3 e^{-\lambda z} \frac{y^2}{2}
\end{aligned}
$$

As it turns out, this is the PDF of an Erlang distribution of order 3. We note that the PDF of an Erlang distribution of order k (meaning there are k exponentials) is the following:

$$
\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}
$$

3. Take $N \sim \textsc{Geom}(p)$ for some constant p. Compute the PDF of $X = \sum_{i=1}^N X_i$. What do you observe? Directly apply the PDF of the Erlang distribution.

$$
\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}
$$

Solution: For the last step, note that the Taylor series expansion about 0 is $e^x = \sum_i \frac{x^{i-1}}{(i-1)!}$.

$$
\begin{aligned}
f_X(x) &= E[f_{X|N}(x)] \\
&= E[\frac{\lambda^N x^{N-1}}{(N-1)!} e^{-\lambda x}] \\
&= \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1}}{(i-1)!} e^{-\lambda x} (1-p)^{i-1} p \\
&= e^{-\lambda x} p \lambda \sum_{i=1}^{\infty} \frac{(\lambda x(1-p))^{i-1} x^{i-1}}{(i-1)!} \\
&= e^{-\lambda x} p \lambda e^{\lambda x(1-p)} \\
&= (\lambda p) e^{-(\lambda p) x}
\end{aligned}
$$

Note that this is another exponential distribution with parameter λp.