Quiz 20 : Markov Chain Hitting Time Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam.

1 Hitting Time

1. Courtney is rushing to class. With probability $\frac{1}{4}$ she takes a step in 3 seconds. With probability $\frac{1}{2}$, she takes 2 steps in 3 seconds. Otherwise, she stops to catch her breath for 3 seconds. How long will it take Courtney to walk 3 steps?

   Solution: Let $\beta(i)$ be the amount of time it takes for Courtney to climb the stairs given she has climbed i steps.

   $$\begin{aligned}
   \beta(3) &= 0 \\
   \beta(2) &= \frac{1}{4}\beta(2) + \frac{3}{4}\beta(3) + 3 \\
   \beta(1) &= \frac{1}{4}\beta(1) + \frac{1}{4}\beta(2) + \frac{1}{2}\beta(3) + 3 \\
   \beta(0) &= \frac{1}{4}\beta(0) + \frac{1}{4}\beta(1) + \frac{1}{2}\beta(2) + 3
   \end{aligned}$$

   Let us plug this into matrix form.

   $$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3/4 & 3/4 & -3 \\ 0 & -3/4 & 1/4 & 1/2 & -3 \\ -3/4 & 1/4 & 1/2 & 0 & -3 \end{bmatrix}$$

   Plugging into Wolfram Alpha, we get the following.

   $$\begin{bmatrix} 1 & 0 & 0 & 0 & 76/9 \\ 0 & 1 & 0 & 0 & 16/3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

   As a result, $\beta(0) = \frac{76}{9}$.

2. Derek is counting his medals; he takes 1 second to place a medal in the "counted" stack. With probability p, he remembers his count. Otherwise, he starts over. How long does it take for him to count n medals? Approximate $1 - p$ to be 0.

Solution: Let $\beta(i)$ denote the expected amount of time it takes for Derek to count the remaining i medals.

$$\beta(i) = p(\beta(i-1) + 1) + (1-p)\beta(n)$$
$$\beta(i) = p\beta(i-1) + (p + (1-p)\beta(n))$$

Let $\alpha = p$ and $\eta = p + (1-p)\beta(n) \approx p$. So,

$$\beta(i) = p^i\beta(0) + \frac{1-\eta^{i-1}}{1-\eta} = \frac{1-\eta^{i-1}}{1-\eta}$$

We are interested in $\beta(n)$, so plug it in and solve.

$$\begin{aligned}\beta(n) &= \frac{1-\eta^{i-1}}{1-\eta} \\ &= \frac{1-p^{i-1}}{1-p}\end{aligned}$$