Quiz 18 : Conditional Expectation Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam.

1 Concepts

1. For each of the following, identify "random variable" or "scalar value". Consider all upper case letters to be random variables and all lower case letters to be scalars.

   (a) $E[Y]$

   Solution: Scalar

   (b) $E[E[Y]]$

   Solution: Scalar

   (c) $E[Y|X]$

   Solution: Random variable

   (d) $E[E[Y|X]]$

   Solution: Scalar

   (e) $\Pr(X|Y)$

   Solution: Scalar

   (f) $E[X^2|Y]$

   Solution: Random variable

   (g) $E[Y|X=x]$

   Solution: Scalar

2. (**True** or **False**) If X and Y are dependent, $cov(X,Y)$ is non-zero.

   Solution: False. We only know the converse of this statement. Counterexample is a 2x2 square centered at the origin.

2 Dilution and Mixing

- Allen is writing an essay for graduate school applications. After writing a page, he decides with probability p to extend his essay's desired length by $\frac{N}{2}$ or with probability $1-p$ to shorten his essay's desired length by $\frac{N}{2}$, where N is the number of desired pages at the moment he makes his decision. Let Allen's initial goal, before he has written any pages, be x pages.

  1. After writing m pages, how many pages does he desire for his essay?

     Solution: First, we will model a single time step. Let X_i be the number of pages Allen intends to write, as he writes the ith page. Then, we can express $E[X_i|X_{i-1}]$ in terms of X_{i-1}.
     With probability p, his essay will be extended by $\frac{X_{i-1}}{2}$ and with probability $1-p$, his essay will be shortened by $\frac{X_{i-1}}{2}$.

$$\begin{aligned} E[X_i|X_{i-1}] &= p(X_{i-1} + \frac{X_{i-1}}{2}) + (1-p)(X_{i-1} - \frac{X_{i-1}}{2}) \\ &= p\frac{3}{2}X_{i-1} + (1-p)\frac{1}{2}X_{i-1} \\ &= \frac{1+2p}{2}X_{i-1} \end{aligned}$$

     We can write out a few iterations to find that

$$\begin{aligned} E[X_i|X_{i-2}] &= (\frac{1+p}{2})^2 X_{i-1} \\ E[X_i|X_{i-3}] &= (\frac{1+p}{2})^3 X_{i-1} \\ &\vdots \\ E[X_i|X_0] &= (\frac{1+2p}{2})^i X_0 \end{aligned}$$

     By the law of iterated expectations, we know that $E[X_i] = E[E[X_i|X_0]]$. We also plug in $X_0 = x$ to get the following.

$$E[X_m] = E[E[X_m|X_0 = x]] = (\frac{1+2p}{2})^m x$$

2. After m pages, how many *remaining* pages do we expect Allen to have to write? Now, N from the original problem is the number of *remaining* pages Allen has to write.

Solution: We will now re-model X_i. Let X_i be the *remaining* number of pages Allen intends to write, as he writes the ith page. Then, we can express $E[X_i|X_{i-1}]$ in terms of X_{i-1}. On the ith page, he has 1 fewer page to write, regardless of his new desired number of pages. We simply take the result from our previous part and subtract 1.

$$E[X_i|X_{i-1}] = \frac{1+2p}{2} X_{i-1} - 1$$

We can again expand this to get the following results. Let $\alpha = \frac{1+2p}{2}$ and $\beta = -1$.

$$\begin{aligned}
E[X_i|X_{i-1}] &= \alpha X_{i-1} + \beta \\
E[X_i|X_{i-2}] &= \alpha(\alpha X_{i-2} + \beta) + \beta \\
&= \alpha^2 X_{i-2} + \alpha\beta + \beta \\
E[X_i|X_{i-3}] &= \alpha^2(\alpha X_{i-3} + \beta) + \alpha\beta + \beta \\
&= \alpha^3 X_{i-3} + \alpha^2\beta + \alpha\beta + \beta \\
&\vdots \\
E[X_i|X_0] &= \alpha^i X_1 + \beta \sum_{j}^{i-1} \alpha^j
\end{aligned}$$

By definition, we have that

$$\sum_{j}^{i} \alpha^j = \frac{1-\alpha^i}{1-\alpha}$$

We can then solve, using the law of iterated expectations.

$$E[X_m] = E[E[X_m|X_0 = x]] = \alpha^m x + \beta \frac{1-\alpha^m}{1-\alpha}$$

Plugging in $\beta = -1$ and leaving $\alpha = \frac{1+2p}{2}$, we then have the following expression.

$$E[X_m] = \alpha^m x - \frac{1-\alpha^m}{1-\alpha}$$