Quiz 13 : Independence, Bayes Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam.

1 Independence and Bayes' Rule

1. Construct a sample space and events A, B, and C so that these events are pairwise independent but not mutually independent.

   Solution: See Crib 13 for definitions of mutual and pairwise independence if need be. We can construct a sample space $\Omega = \{1, 2, 3, 4\}$. Let $A = \{1, 2\}, B = \{2, 3\}, C = \{1, 3\}$. The probability of $\Pr(A|B) = \Pr(A)$ for all pairs. $\Pr(A, B, C) = 0$, whereas $\Pr(A)\Pr(B)\Pr(C) = \frac{1}{8} \neq 0$.

2. Is it possible to compute $\Pr(B|A)$ given $\Pr(A|B)$, $\Pr(B)$, and $\Pr(\bar{A}|\bar{B})$? If so, compute it.

   Solution: Yes.

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A|B)\Pr(B) + \Pr(A|\bar{B})\Pr(\bar{B})}$$

   Note that $\Pr(A|\bar{B}) + \Pr(\bar{A}|\bar{B}) = 1$, and $\Pr(B) + \Pr(\bar{B}) = 1$, so we can solve for $\Pr(A|\bar{B})$ and $\Pr(\bar{B})$.

Consider a fair coin c_1 and a coin c_2 with bias p. Roll a 6-sided dice. If you roll 1 or 2, flip c_1. If you roll 3, flip either c_1 or c_2 with probability $\frac{1}{2}$. If you roll 4, 5, or 6, flip c_2.

1. What is the probability that you rolled a 3, given you see heads?

   Solution: Let A be the event you rolled 1 or 2, let B be the probability you roll a 3, let C be the probability that you roll a 4, 5, or 6, and let H be the probability that you roll a heads.

$$\Pr(B|H) = \frac{\Pr(H|B)\Pr(B)}{\Pr(H|A)\Pr(A) + \Pr(H|B)\Pr(B) + \Pr(H|C)\Pr(C)}$$

We will first reason about the numerator. We know that the $\Pr(B) = \frac{1}{6}$. Given that we've rolled a 3, we know that we have $\frac{1}{2}$ probability of getting either the fair coin c_1 or the biased coin c_2. Thus, the probability of heads is

$$\begin{aligned}
\Pr(H|B) &= \Pr(H, c_1, B) + \Pr(H, c_2, B) \\
&= \Pr(H|c_1, B)\Pr(c_1|B) + \Pr(H|c_2, B)\Pr(c_2|B) \\
&= \frac{1}{2}\frac{1}{2} + p\frac{1}{2} \\
&= \frac{1}{2}(\frac{1}{2} + p)
\end{aligned}$$

This makes the numerator

$$\Pr(H|B)\Pr(B) = \frac{1}{2}(\frac{1}{2} + p)\frac{1}{6} = \frac{1}{12}(\frac{1}{2} + p)$$

We can compute the first term in the denominator. Note that for the event A (where we roll a 1 or 2), we are guaranteed to flip the fair coin. Additionally, there is a $\frac{2}{6} = \frac{1}{3}$ probability of flipping either a 1 or a 2.

$$\Pr(H|A)\Pr(A) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$$

The third term in the denominator is given by the following. Note that the probability of rolling a 4, 5, or 6 is $\frac{3}{6} = \frac{1}{2}$.

$$\Pr(H|C)\Pr(c) = p\frac{1}{2}$$

We have the denominator is thus the following.

$$\Pr(H) = \frac{1}{6} + \frac{1}{12}(\frac{1}{2} + p) + \frac{p}{2} = \frac{14p + 5}{24}$$

Our final expression is the following.

$$\begin{aligned}
\Pr(B|H) &= \frac{\Pr(H|B)\Pr(B)}{\Pr(H|A)\Pr(A)+\Pr(H|B)\Pr(B)+\Pr(H|C)\Pr(C)} \\
&= \frac{\frac{1}{12}(\frac{1}{2}+p)}{\frac{14p+5}{24}} \\
&= \frac{1+2p}{5+14p}
\end{aligned}$$

2. What is the probability that you rolled 3 or fewer, given you see heads?

Solution: Let X_1 be the event that we roll 1. Let X_2 be the probability that we roll a 2. We are currently looking for $\Pr(X_{(1,2,3)}|H) = \Pr(X_1|H) + \Pr(X_2|H) + \Pr(X_3|H)$.

We can take $\Pr(H)$ from the previous part, and we know that rolling a 1 (X_1) will definitely give us a fair coin, with $\frac{1}{2}$ probability of being heads.

$$\begin{aligned}
\Pr(X_1|H) &= \frac{\Pr(H|X_1)\Pr(X_1)}{\Pr(H)} \\
&= \frac{\frac{1}{2}\frac{1}{6}}{\frac{14p+5}{24}} \\
&= \frac{2}{14p+5}
\end{aligned}$$

We know that X_2 will have the same probability, since the probability of rolling a 2 is the same as rolling a 1. Thus, the final answer is

$$\begin{aligned}
\Pr(X_{(1,2,3)}|H) &= 2\Pr(X_1|H) + \Pr(X_3|H) \\
&= \frac{4}{14p+5} + \frac{1+2p}{5+14p} \\
&= \frac{5+2p}{14p+5}
\end{aligned}$$

3. What is the probability that you rolled more than 3 given that you see heads?

Solution: Note that $\Pr(X_1|H)+\Pr(X_2|H)+\cdots+\Pr(X_6|H) = 1$, thus $\Pr(X_{(1,2,3)|H} + \Pr(X_{(4,5,6)}|H) = 1$, and we have that $\Pr(X_{(4,5,6)} = 1 - \Pr(X_{(1,2,3)}|H)$. This is simply one minus the answer to part 2.

$$1-\frac{5+2p}{14p+5}=\frac{12p}{14p+5}$$