Quiz 12 : Probability Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam.

1 Probability and Conditional Probability

Consider a 5-card hand from a standard deck of 52 cards.

1. What is the probability that we have exactly 1 ace?

   Solution: It is easier to reason about this using counting. We pick an ace and then we pick from the non-Ace cards.

   $$\frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$$

   Alternatively, we can consider the possibility of picking an Ace $\frac{4}{52}$ and then picking four non-Ace cards, $\frac{48}{51}\frac{47}{50}\frac{46}{49}\frac{45}{48}$. Note that this assumes we pick in this order, so we must divide by the ways to order these cards; there are 5! ways to order 5 cards, so we have

   $$\frac{4 * 48 * 47 * 46 * 45}{52 * 51 * 50 * 49 * 48 * 5!}$$

2. What is the probability that we have exactly 3 clubs? (Hint: Use counting)

   Solution: It is easier to reason about this using counting. We pick our three clubs. Since there are 13, we choose 3 from 13, and since there are 52 - 13 = 39 non-club cards, we pick 2 from 39.

   $$\frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}}$$

3. Given we have three clubs, what is the probability of an ace of clubs?

   Solution: In the denominator, we count all ways to pick three clubs, which is the numerator from part 2.

$$\binom{13}{3}\binom{39}{2}$$

   In the numerator, we count all ways to pick a 5-card hand with an ace of clubs. Given that our ace is a club, we only have two more clubs to choose and two non-clubs to choose.

$$\binom{12}{2}\binom{39}{2}$$

   Thus, our final answer is the following.

$$\frac{\binom{12}{2}\binom{39}{2}}{\binom{13}{3}\binom{39}{2}} = \frac{3}{13}$$

4. What is the probability that we have 3 clubs or 1 ace? (Not XOR, Hint: Think about inclusion-exclusion.)

   Solution: We need to apply inclusion-exclusion. Recall that this states for two events A and B,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

   We have that A is from part 1 and B is from part 2. All that remains is to compute $\Pr(A \cap B)$ which is the probability of 3 clubs *and* exactly one ace. We have two cases; either the ace is a club or the ace is not a club. Let C be the event that the ace is a club. By some variant of the law of total probability:

$$\Pr(A \cap B) = \Pr(A \cap B \cap C) + \Pr(A \cap B \cap \bar{C})$$

   We now compute both probabilities. Pick the ace of clubs, $\binom{1}{1}$. We pick our two non-ace clubs from 12 non-ace clubs $\binom{12}{2}$. Finally, we pick our non-ace, non-club from $52 - 4 - 13 + 1 = 36$.

$$\Pr(A \cap B \cap C) = \frac{\binom{12}{2}\binom{36}{2}}{\binom{52}{5}}$$

Given that our ace is *not* a club, we have all three non-Ace clubs to choose ($\binom{12}{3}$), not to mention a non-club, non-ace and a non-club ace card. We know there are $52 - 13 - 3 = 36$ non-club, non-ace cards and there are 3 non-club, ace cards. Thus, we have the following.

$$\Pr(A \cap B \cap \bar{C}) = \frac{\binom{12}{3}\binom{36}{1}\binom{3}{1}}{\binom{52}{5}}$$

Thus, we can combine these to get the following for $\Pr(A \cap B)$.

$$\frac{\binom{12}{2}\binom{36}{2}}{\binom{52}{5}} + \frac{\binom{12}{3}\binom{36}{1}\binom{3}{1}}{\binom{52}{5}}$$

We then have our final expression for $\Pr(A \cup B)$.

$$\frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} + \frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} - (\frac{\binom{12}{2}\binom{36}{2}}{\binom{52}{5}} + \frac{\binom{12}{3}\binom{36}{1}\binom{3}{1}}{\binom{52}{5}})$$