Quiz 4 Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam.

1 Graph Theory

For each of the following, consider a graphs without self-loops or multi-edges. For each True or False question, provide a brief justification.

1. (True or False) The sum of all degrees in an undirected *tree* with n vertices is $2(n-1)$.

   Solution: True

   For a tree with n vertices, we have $n-1$ edges. The sum of all degrees is twice the number of edges, making $2(n-1)$.

2. (Short Answer) Take a binary tree with n vertices, where for some $k \in \mathbb{Z}$, $n = \sum_i^k 2^i$. (i.e., enough vertices for a complete tree) Find the longest path, and remove all of its edges. The resulting graph has *at most* how many degrees?

   Solution: $2(n-1) - 4\lfloor \log n \rfloor$

   We want to compute the maximum number of degrees, meaning the maximum number of edges. To achieve this, we need to remove the fewest number of edges, when we remove the edges in the longest path. We can minimize the length of the longest path by considering a complete, balanced binary tree. For n vertices, the tree has height $\lfloor \log n \rfloor$. The longest path is thus $2\lfloor \log n \rfloor$. From part 1, the sum of degrees in an undirected tree with n vertices is $2(n-1)$. Since we removed $2\lfloor \log n \rfloor$ edges, we removed $4 \log n$ degrees, making $2(n-1) - 4\lfloor \log n \rfloor$ the maximum sum of degrees.

3. Given a tree with undirected edges, prove that each vertex u has exactly one path to any other vertex v. You may use any of the other definitions of a tree.

   Solution: Assume for contradiction that this is not true. This means for some vertex u, there are at least two paths to another vertex v. Let two of these paths be p_1 and p_2. The first vertex where they differ is u', and the first vertex where they again share a vertex after u' is v'. Since we can take a path from u' to v' and then back to v' without repeating vertices, we have constructed a cycle starting and ending at u'. A tree by definition does not contain cycles. Contradiction.

Thus, the provided statement is true.

Why could we not just go from u to v and back, making a cycle?

A cycle cannot repeat vertices. Since the two paths between u and v, p_1 and p_2 respectively, are not guaranteed to be vertex-disjoint (i.e., not guaranteed to not share vertices), we cannot claim those two paths form a cycle.