Quiz 2 Solutions

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This quiz does not count towards your grade. It exists to simply gauge your understanding. Treat this as though it were a portion of your midterm or final exam. In this quiz, we will walk through several misconceptions.

1 Proofs

Prove or disprove each of the following statements.

1. Prove that on an 8×8 chess board, there cannot exist more than 8 "peaceful" rooks. Two rooks are "peaceful" if they do not share a common row or column. A set of rooks is peaceful if all pairs in the set are peaceful.

   Solution: Assume for contradiction we can have 9 or more peaceful rooks in one dimension. We know that this dimension has only 8 slots. If each slot holds at most 1 rook, by the pigeonhole principle, this dimension holds at most 8 rooks. Contradiction.

   We can consider the same proof in both dimensions. If this is true for all columns *and* this is true for all rows, this is true for our 8×8 board.

2. We now generalize this to an n-dimensional chess board, where all dimensions have 8 slots. Prove that there still cannot exist more than 8 "peaceful" rooks, in any $n \in \mathbb{Z}^+$ dimension. (The chess board has 2 dimensions).

   Solution: We induct over the dimensions.

   Base Case: The proof of the first dimension is problem in 1.

   Inductive Hypothesis: Assume that in the k-th dimension, where each dimension has 8 slots, there cannot exist more than 8 "peaceful" rooks.

   Inductive Step: Consider the k+1 dimension space. Reducing the dimensionality of this space so that it is now k-dimensional, we can place at most 8 "peaceful" rooks. Now, we again reconsider our k+1th dimension.

   This dimension contains only 8 slots, and by the pigeonhole principle, this means we can have at most 8 "peaceful" rooks along the new, k+1th dimension. As a result, we can only have 8 "peaceful" rooks in the k+1th dimension, in general.