Crib 16 : Inequalities

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The crib sheet contains cheat-sheet worthy information but is not a substitute for lectures or for reading the notes. It also contains pointers and common mistakes.

1 Material

- Markov's inequality states the following. Remember that $\alpha > 0$. (Derivation below.)

$$\Pr(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

- Chebyshev's inequality states the following. Intuitively, it is the probability that we are *more* than some distance α from the mean. Remember that $\alpha > 0$. (Derivation below.)

$$\Pr(|X - \mu| \geq \alpha) \leq \frac{\text{var}(X)}{\alpha^2}$$

- Consider some random variable X, its mean μ, some positive α, and probability p. A **confidence interval** is an interval of α distance from μ that we know X has probability p of falling in.

  - Remember the distinction between observable values and non-observable values. See quiz 16 for practice. We usually have some p that we'd like to estimate but cannot observe directly. As a result, we observe some q, express it in terms of p and then solve for p. This is then our estimate for p.

- The **Law of Large Numbers** states that as $n \to \infty$, our estimate of the mean approaches the true mean. More formally, $\Pr(|A_n - \mu| \geq \alpha) \to 0$, where $A_n = \frac{\sum_{i=1}^n X_i}{n}$

2 Short Proofs

2.1 Markov's

This is a one-line derivation, using expectation.

$$E[X] = \sum a \Pr(a) \geq \sum_{a>\alpha} a \Pr(a) \geq \alpha \sum_{a>\alpha} \Pr(a) \geq \alpha \Pr(a \geq \alpha)$$

We can then re-arrange to obtain our final result $\Pr(a \geq \alpha) \leq \frac{E[X]}{\alpha}$.

2.2 Chebyshev's

Consider Markov's for $Y = (X - \mu)^2$. Note that $(X - \mu)^2 > \alpha^2$ is the same as $|X - \mu| > \alpha$, and recall that $\text{var}(X) = E[(X - \mu)^2]$.

$$\begin{aligned}
\Pr(Y \geq \alpha) &\leq \frac{E[Y]}{\alpha} \\
\Pr(X - \mu \geq \alpha) &\leq \frac{E[X - \mu]}{\alpha} \\
\Pr((X - \mu)^2 \geq \alpha^2) &\leq \frac{E[(X - \mu)^2]}{\alpha^2} \\
\Pr(|X - \mu| \geq \alpha) &\leq \frac{}{\alpha^2}
\end{aligned}$$