Crib 15 : Expectation, Variance

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The crib sheet contains cheat-sheet worthy information but is not a substitute for lectures or for reading the notes. It also contains pointers and common mistakes.

1 Expectation

- Expectation of a random variable X is intuitively, the mean

$$E[X] = \sum x \Pr(X = x)$$

- Don't try to rationalize the following; taken as a whole, there is no intuitive meaning for $g(X)$.

$$E[g(X)] = \sum g(x) \Pr(X = x)$$

- The linearity of expectation **always holds**, regardless of the independence (or lack thereof). For example,

$$E[2X + 3Y] = 2E[X] + 3E[Y]$$

More generally, take random variables X_i and constants α_i.

$$E[\sum_i \alpha_i X_i] = \sum_i \alpha_i E[X_i]$$

2 Covariance

- Take two random variables X, Y. They do not need to be independent. We have the following expression. See 3.2 for a proof.

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

- Covariances sum, like vectors, and are symmetric.

$$\text{cov}(X, Y) = \text{cov}(Y, X)$$
$$\text{cov}(A + B, Y) = \text{cov}(A, Y) + \text{cov}(B, Y)$$

3 Variance

- Variance of a random variable X is intuitively, the squared distance from the mean, or the *spread*. See 3.1 for a proof.

$$\text{var}(X) = E[X^2] - E[X]^2$$

- Variance is unaffected by a constant shift, α. Remember variance is the spread of our X.

$$\text{var}(X + \alpha) = \text{var}(X)$$

- A scalar constant is squared when taken out of variance.

$$\text{var}(\alpha X) = \alpha^2 \text{var}(X)$$

- Take two random variables X, Y. They do not need to be independent. We have the following expression.

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$$

- Here's a nifty trick: Say you know variance and the expected value of a random variable X. Then, we can compute $E[X^2]$ fairly efficiently! As a matter of fact, it is $E[X^2] = \text{var}(X) + E[X]^2$

4 Independence

- X, Y are independent *if and only if* $\Pr(X)\Pr(Y) = \Pr(X, Y)$
- If X, Y are independent, then $E[X]E[Y] = E[XY]$, but the converse is not necessarily true.
- If X, Y are independent, then $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$, but the converse is not necessarily true. We call this the *linearity of variance*, but remember that this holds only if X, Y are independent!
- If X, Y are independent, then $\text{cov}(X, Y) = 0$, but the converse is not necessarily true.

5 Short Derivations

You are not responsible for the proofs and derivations in the section, but they may be enlightening.

5.1 Variance

We can derive this in a few lines. We use the law of iterated expectations below, $E[E[X]] = E[X]$, which won't be introduced until later in the course.

$$\begin{aligned}
\text{var}(X) &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E[X]^2] \\
&= E[X^2] - 2E[X]^2 + E[X]^2 \\
&= E[X^2] - E[X]^2
\end{aligned}$$

5.2 Covariance

We again apply the law of iterated expectation.

$$
\begin{aligned}
\operatorname{cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY - 2E[X]E[Y] + E[X]E[Y]] \\
&= E[XY] - 2E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$