Quiz 3

03 Support Vector Machines, Convex Optimization

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1 Convexity

Prove that if $f(x)$ is convex, $f(\alpha x + \beta)$ is convex for scalars α, β. Hint: If you're stuck, take $g(x) = \alpha x + \beta$.

Solution: Recall that a function is convex if

$$\forall x_1, x_2 \in \mathbb{R}, t \in [0,1], f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$$

Take $g(x) = \alpha x + \beta$ and the following $f(g(x))$; we first prove a lemma.

$$\begin{aligned}
&f(g((1-t)x_1 + tx_2)) \\
&= f(\alpha((1-t)x_1 + tx_2) + \beta) \\
&= f((1-t)\alpha x_1 + (1-t)\beta + t\alpha x_2 + t\beta) \\
&= f((1-t)(\alpha x_1 + \beta) + t(\alpha x_2 + \beta)) \\
&= f((1-t)g(x_1) + tg(x_2))
\end{aligned}$$

Then, simply apply the convexity of f.

$$\begin{aligned}
f(g((1-t)x_1 + tx_2)) &= f((1-t)g(x_1) + tg(x_2)) \\
&\leq (1-t)f(g(x_1)) + tf(g(x_2))
\end{aligned}$$

2 Linear Algebra

Compute the variance of $u \in \mathbb{R}^n$, where $u \sim (0, I)$. This notation simply means that u is sampled from some distribution with mean 0, where the covariance matrix of u is I. Consider $A \in \mathbb{R}^{n \times n}$. Compute variance of $y = Au$.

Solution:

$$\begin{aligned}
&E[(Au - \mu)^T (Au - \mu)] \\
&= E[(Au)^T Au] \\
&= E[u^T A^T Au] \\
&= E[Tr(A^T Auu^T)] \\
&= Tr(E[A^T Auu^T]) \\
&= Tr(A^T A) \\
&= \|A\|_F^2
\end{aligned}$$

As it turns out, this is precisely the Frobenius norm.