Quiz 1 Solutions

01 Background

by Alvin Wan . alvinwan.com/cs189/fa17

Treat this as an exam situation. You will be given 5 minutes to complete this quiz.

1 PSD-ness

Consider a real, symmetric matrix A.

(a) Prove that if all of its eigenvalues are non-negative, $\exists B$ s.t. $A = BB^T$.

Solution: If all $\lambda_i \geq 0$, then $\forall i$, $\sqrt{\lambda_i}$ is well-defined. Let $\Lambda^{1/2}$ denote a diagonal matrix where its non-zero entries are $\sqrt{\lambda_i}$. Note additionally that A is real, symmetric and is thus diagonalizable. Then,

$$A = P\Lambda P^T = P\Lambda^{1/2}\Lambda^{1/2}P^T = (P\Lambda^{1/2})(P\Lambda^{1/2})^T = BB^T$$

such that $B = P\Lambda^{1/2}$.

(b) Prove that if $\exists B$ s.t. $A = BB^T$, then the quadratic form is non-negative $\forall x, x^TAx \geq 0$.

Solution:

$$\forall x, x^TAx = x^TBB^Tx = (B^Tx)^T(B^Tx) = \|B^Tx\|_2^2 \geq 0$$

The last step is true, because all norms are non-negative by construction.

In our discussion worksheet, we will then prove that $\forall x, x^TAx \geq 0$ implies A's eigenvalues are all non-negative! This proves that these three conditions are all equivalent conditions for PSD-ness of a matrix A.